4. The maximum angle at which a body can repose (sleep) is called *Angle of repose*. This is equal to the angle of friction.
5. The inverted cone with semicentral angle equal to the angle of friction is called *Cone of friction*.

Important Formulae

1. $\mu = \frac{F}{N}$, where F is limiting friction, N is normal reaction and μ is coefficient of friction.
2. $\theta = \phi = \tan^{-1}\mu$, where θ is angle of friction, ϕ is angle of repose and μ is coefficient of friction.

Questions

Theory:

1. State the laws of dry friction.
2. Define the terms: (i) Coefficient of friction, (ii) Angle of friction and (iii) Cone of friction.

Problems

1. A pull of 180 N applied upward at 30° to a rough horizontal plane was required to just move a body resting on the plane while a push of 2000 N applied along the same line of action was required to just move the same body downwards. Determine the weight of the body and the coefficient of friction.

   [**Ans.** W = 990 N; μ = 0.1722]
2. The block A shown in Fig. 10.16 weights 2000 N. The cord attached to A passes over a frictionless pulley and supports a weight equal to 800 N. The value of coefficient of friction between A and the horizontal plane is 0.35. Solve for horizontal force P: (1) if motion is impending towards the left, and (2) if the motion is impending towards the right.

   [**Ans.** (1) 1252 N; (2) 132.82 N]

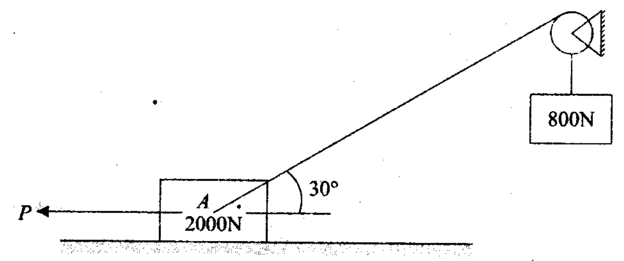


Fig. 10.16

3. A 3000 N block is placed on an inclined plane as shown in Fig. 10.17. Find the maximum value of W for equilibrium if tipping does not occur. A ssume coeficient of friction as 0.2. [**Ans.** 2636.15]

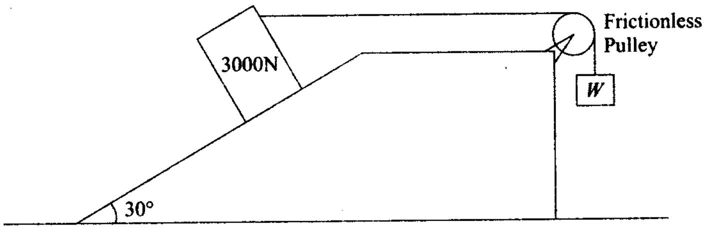


Fig. 10.17

4. Find whether block A is moving up or down the plane in Fig. 10.18 for the data given below. Weight of block A = 300 N. Weight of block B = 600 N. Coefficient of limiting friction between plane AB and block A is 0.2. Coefficient of limiting friction between plane BC and block B is 0.25. Assume pulley as smooth.

[**Ans.** The block A is stationary sin F developed < F_{min}]

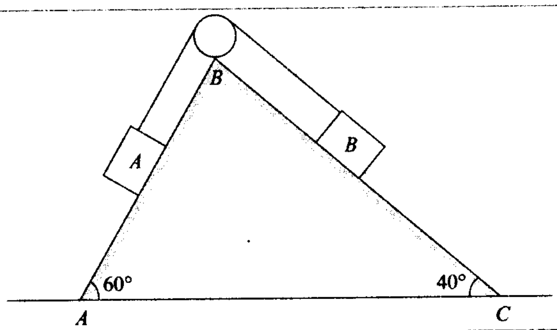


Fig. 10.18

5. Two identical blocks A and B are connected by a rod and they rest against vertical and horizontal planes respectively as shown in Fig. 10.19. If sliding impends when θ = 45°, determine the coefficient of friction, assuming it to be the same for both floor and wall.

[**Ans.** 0.414]

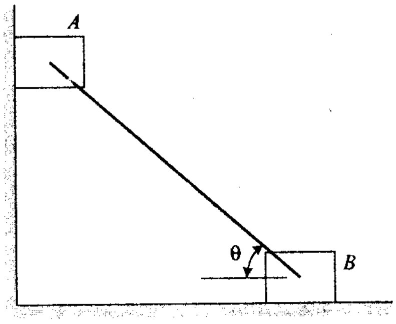


Fig. 10.19

6. Determine the force P required to start the wedge as shown in Fig. 10.20. The angle of friction for all surfaces of contact is 15°.

[Ans. 26.6784 kN]

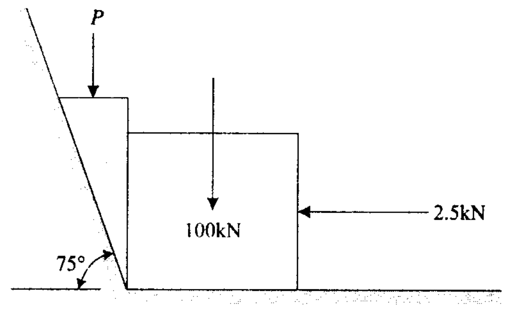


Fig. 10.20

7. Two blocks A and B weighing 15 kN and 3 kN, respectively, are held in position against an inclined plane by applying a horizontal force P as shown in Fig. 10.21. Find the least value of P which will include motion of the block A upwards. Angle of friction for all contact surfaces if 12°.

[Ans. 9.69 kN]

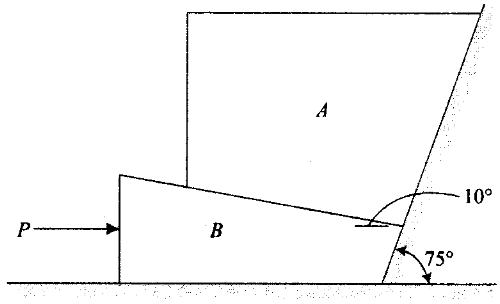


Fig. 10.21

8. Find the horizontal force P required to push the block A of weight 150 N which carries block B of weight 1280 N as shown in Fig. 5.22. Take angle of limiting friction between floor and block A as 14° and that between vertical wall and block B as 13° and coefficient of limiting friction between the blocks as 0.3.

[Ans. P = 1331.6 N]

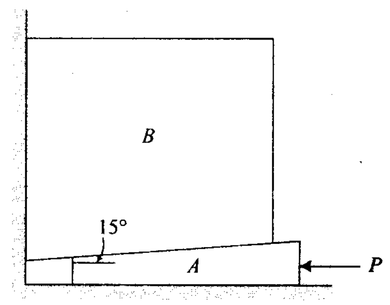


Fig. 10.22

9. The level of a precast beam weighing 20,000 N is to be adjusted by driving a wedge as shown in Fig. 10.23. If coefficient of friction between the wedge and pier is 0.35 and that between beam and the wedge is 0.25, determine the minimum force P required on the wedge to make adjustment of the beam. Angle of the wedge is 15°.

   (Hint: Vertical component of reaction on wedge at contact with beam = 1/2 vertical load on beam = 10,000 N).

   [**Ans.** 9057.4 N]

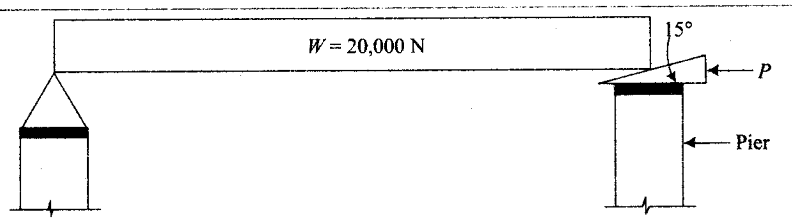


Fig. 10.23

10. A ladder 5 m long rests on a horiontal ground and leans against a smooth vertical wall at an angle of 70° with the horizontal. The weight of the ladder is 300 N. The ladder is on the verge of sliding when a man weighing 750 N stands on a rung 1.5 m high. Calculate the coefficient of friction between the ladder and the floor.

    [**Ans.** μ = 0.1837]

11. A 4 m ladder weighing 200 N is placed against a vertical wall is shown in Fig. 10.24. As a man weighing 800 N, reaches a point 2.7 m from A, the ladder is about to slip. Assuming that the coefficient of friction between the ladder and the wall is 0.2, determine the coefficient of friction between the ladder and the floor.

    [**Ans.** 0.3548]

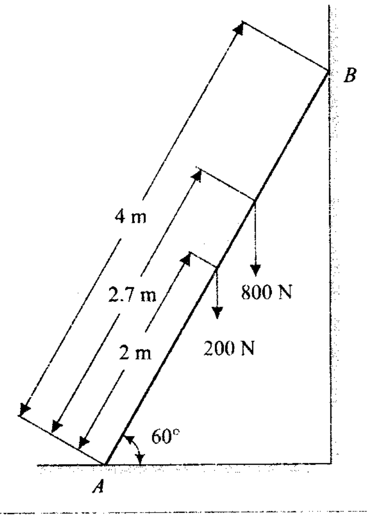


Fig. 10.24

CHAPTER 11

Moment of Inertia of an Area

Moment of inertia of an area is another important property of cross-sections of the members like beam column and many mechanical components. Determination of moment of inertia of the plane areas is of great importance in the study of subjects like strength of materials, structural designs and machine designs. In this chapter the terms moment of inertia, polar moment of inertia and radius of gyration are defined and explained. The two important theorems, namely, perpendicular axis theorem and parallel axis theorem are stated and derived. Then derivation of expressions for simple figures like rectangle, triangle and circular sections are carried out by method of integration. Using these standard expressions, moment of inertia of composite areas is evaluated for a number of sections.

11.1 MOMENT OF INERTIA OF PLANE FIGURE

Consider the area shown in Fig. 11.1(a). dA is an elemental area with coordinates as x and y. Then the term $\Sigma y^2 dA$ is called *moment of inertia* of the area about x-axis and is denoted by I_{xx}. Thus

$$I_{xx} = \sum y^2 dA \qquad \text{Eqn. (11.1)}$$

Similarly, the moment of inertia about y-axis is

$$I_{yy} = \sum x^2 dA \qquad \text{Eqn. (11.2)}$$

In general, if r is the distance of elemental area dA from the axis AB [Ref. Fig. 11.1(b)], the sum of the terms $r^2 dA$ to cover the entire area is called moment of inertia about the axis AB. If r and dA can be expressed in general form for an element in the plane area, then the sum becomes an integration with the suitable limits to cover the entire area. Thus,

$$I_{AB} = \sum r^2 dA = \oint r^2 dA \qquad \text{Eqn. (11.3)}$$

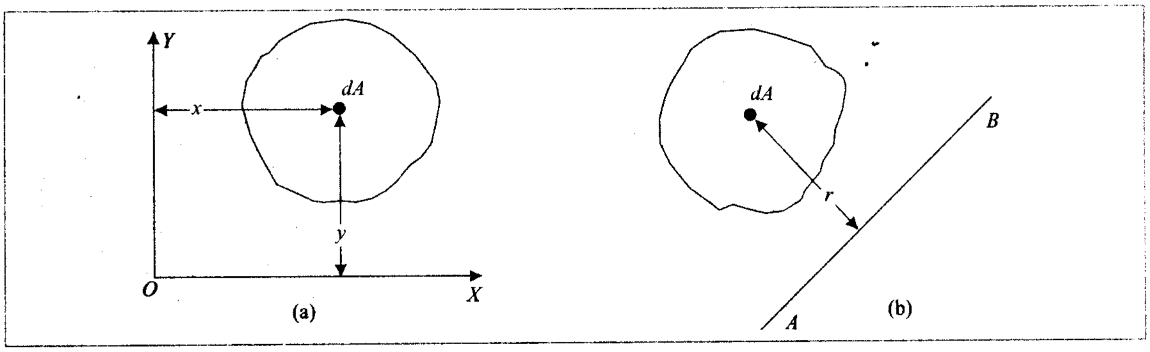


Fig. 11.1

The term rdA may be called as moment of area, a term similar to moment of a force, and hence r^2dA may be called as *moment of moment of area* or the *second moment of area*. Thus, the moment of inertia of a plane figure is nothing but second moment of area. In fact, the term second moment of area appears to correctly signify the meaning of the expression $\Sigma r^2 dA$. The term moment of inertia is rather a misnomer. However, the term moment of inertia has come to stay for long time and hence it will be used in this book also.

Though moment of inertia of plane is a purely mathematical term, it is one of the important properties of areas. The strength of members subject to bending depends on the moment of inertia of its cross-sectional area. Students will find this property of area very useful when they study subjects like strength of materials, structural design and machine design.

The moment of inertia is a fourth order term in linear unit since it is a term obtained by multiplying area by the square of the distance. Hence, in SI units, if metre (m) is the unit for linear measurements used then m^4 is the unit of moment of inertia. If millimeter (mm) is the unit used for linear measurements, then mm^4 is the unit of moment of inertia. In MKS system m^4 or cm^4 and in FPS system ft^4 or in^4 are commonly used as units for moment of inertia.

11.2 POLAR MOMENT OF INERTIA

Moment of inertia about an axis perpendicular to the plane of an area is known as **polar moment of inertia.** It may be denoted as J or I_{zz}. Thus, the moment of inertia about an axis perpendicular to the plane of the area at O in Fig. 11.2 is called polar moment of inertia at point O, and is given by

$$I_{zz} = \sum r^2 dA \qquad \text{Eqn. (11.4)}$$

11.3 RADIUS OF GYRATION

Radius of gyration is a mathematical term defined by the relation

$$k = \sqrt{\frac{I}{A}} \qquad \text{Eqn. (11.5)}$$

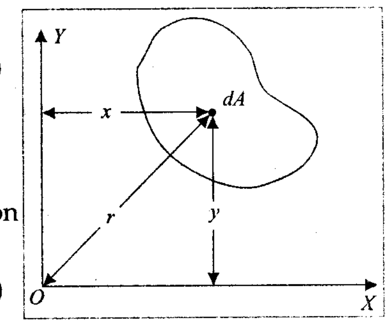


Fig. 11.2

where k = radius of gyration
I = moment of inertia
and A = the cross-sectional area.

Suffixes with moment of inertia I also accompany the term radius of gyration k.

Thus, we can have,

$$k_{xx} = \sqrt{\frac{I_{xx}}{A}}$$

$$k_{yy} = \sqrt{\frac{I_{yy}}{A}}$$

$$k_{AB} = \sqrt{\frac{I_{AB}}{A}}$$

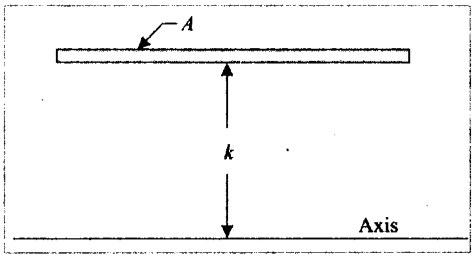


Fig. 11.3

and so on.

The relation between radius of gyration and moment of inertia can be put in the form:

$$I = Ak^2 \qquad \text{Eqn. (11.6)}$$

From the above relation a geometric meaning can be assigned to the term 'radius of gyration'. We can consider k as the distance at which the complete area is squeezed and kept as a strip of negligible width (Fig. 11.3) such that there is no change in the moment of inertia.

11.4 THEOREMS OF MOMENTS OF INERTIA

There are two theorems of moment of inertia:

(1) Perpendicular axis theorem, and
(2) Parallel axis theorem.

These are explained and proved below.

Perpendicular Axis Theorem

The moment of inertia of an area about an axis perpendicular to its plane (polar moment of inertia) at any point O is equal to the sum of moments of inertia about any two mutually perpendicular axes through the same point O and lying in the plane of the area.

Referring to Fig. 11.4, if $z - z$ is the axis normal to the plane of paper passing through point O, as per this theorem,

$$I_{zz} = I_{xx} + I_{yy} \qquad \text{Eqn. (11.7)}$$

The above theorem can be easily proved. let us consider an elemental area dA at a distance r from O. Let the coordinates of dA be x and y. Then from definition:

$$I_{zz} = \sum r^2 dA$$

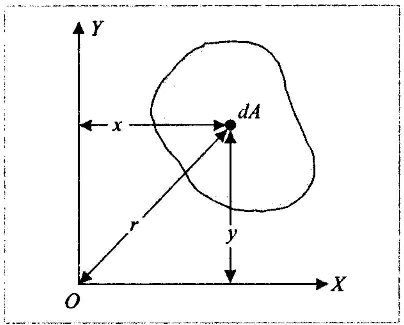


Fig. 11.4

$$= \sum (x^2 + y^2)dA$$

$$= \sum x^2 dA + \sum y^2 dA$$

$$I_{zz} = I_{xx} + I_{yy}, \text{ since } \sum x^2 dA = I_{yy} \text{ and } \sum y^2 dA = I_{xx}.$$

Parallel Axis Theorem

Moment of inertia about any axis in the plane of an area is equal to the sum of moment of inertia about a parallel centroidal axis and the product of area and square of the distance between the two parallel axes. Referring to Fig. 11.5, the above theorem means:

$$I_{AB} = I_{GG} + Ay_c^2 \qquad \text{Eqn. (11.8)}$$

where

I_{AB} – moment of inertia about axis AB,

I_{GG} – moment of inertia about centroidal axis GG parallel to AB,

A – the area of the plane figure given, and

y_c – the distance between the axis AB and the parallel centroidal axis GG.

Proof: Consider an elemental parallel strip dA at a distance y from the centroidal axis (Fig. 11.5).

Then, $$I_{AB} = \sum (y + y_c)^2 dA$$

$$= \sum (y^2 + 2yy_c + y_c^2)dA$$

$$= \sum y^2 dA + \sum 2yy_c dA + \sum y_c^2 dA$$

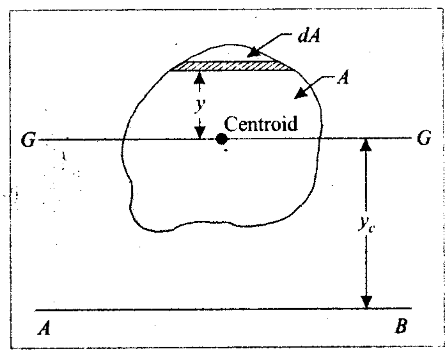


Fig. 11.5

Now,

$$\sum y^2 dA = \text{Moment of inertia about the axis } GG$$

$$= I_{GG}$$

$$\sum 2yy_c dA = 2y_c \sum y dA$$

$$= 2y_c A \frac{\sum y\,dA}{A}$$

In the above term $2y_c A$ is constant and $\frac{\sum y\,dA}{A}$ is the distance of centroid from the reference axis GG. Since GG is passing through the centroid itself $\frac{y\,dA}{A}$ is zero and hence the term $\Sigma 2yy_c dA$ is zero.

Now, the third term,

$$\sum y_c^2 dA = y_c^2 \sum dA$$

$$= A y_c^2$$

$$\therefore \quad I_{AB} = I_{GG} + A y_c^2$$

Note: The above equation cannot be applied to any two parallel axes. One of the axes (GG) must be centroidal axis only.

11.5 MOMENT OF INERTIA FROM METHOD OF INTEGRATION

For simple figures, moment of inertia can be obtained by writing general expression for an element and then carrying out integration so as to cover the entire area. This procedure is illustrated with the following three cases:

(1) Moment of inertia of rectangle about the centroidal axis

(2) Moment of inertia of a triangle about the base

(3) Moment of inertia of a circle about a diametral axis.

Moment of Inertia of a Rectangle about the Centroidal Axis – Consider a rectangle of width b and depth d (Fig. 11.6). Moment of inertia about the centroidal axis $x - x$ parallel to the short side is required.

Consider an elemental strip of width dy at a distance y from the axis. Moment of inertia of the elemental strip about the centroidal axis xx is:

$$= y^2 dA$$

$$= y^2 b dy$$

$$\therefore \quad I_{xx} = \int_{-\frac{d}{2}}^{\frac{d}{2}} y^2 b\, dy$$

$$= b\left[\frac{y^3}{3}\right]_{-\frac{d}{2}}^{\frac{d}{2}}$$

$$= b\left[\frac{d^3}{24} + \frac{d^3}{24}\right]$$

$$I_{xx} = \frac{bd^3}{12}$$

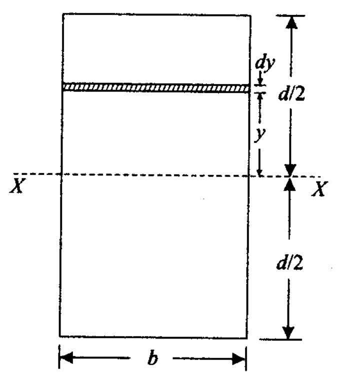
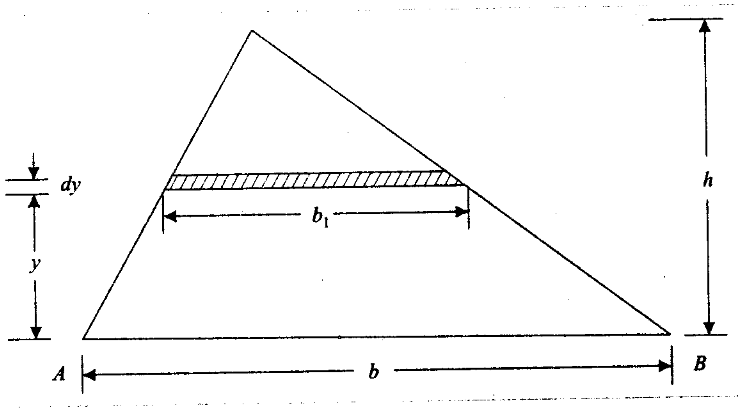


Fig. 11.6

Moment of Inertia of a Triangle about its Base – Moment of inertia of a triangle with base width b and height h is to be determined about the base AB (Fig. 11.7).

Consider an elemental strip at a distance y from the base AB. Let dy be the thickness of the strip and dA its area. Width of this strip is given by:



$$b_1 = \frac{(h-y)}{h} \times b$$

Moment of inertia of this strip about AB

$$= y^2 dA$$
$$= y^2 b_1 dy$$
$$= y^2 \frac{(h-y)}{h} \times b \times dy$$

$\therefore$ Moment of inertia of the triangle about AB,

$$I_{AB} = \int_0^h \frac{y^2(h-y)bdy}{h}$$

$$= \int_0^h \left(y^2 - \frac{y^3}{h}\right) bdy$$

$$= b\left[\frac{y^3}{3} - \frac{y^2}{4h}\right]_0^h$$

$$= b\left[\frac{h^3}{3} - \frac{h^4}{4h}\right]$$

$$I_{AB} = \frac{bh^3}{12}$$

Moment of Inertia of a Circle about its Distance Axis – Moment of inertia of a circle of radius R is required about its diametral axis as shown in Fig. 11.8.

Consider an element of sides $rd\theta$ and dr as shown in the figure. Its moment of inertia about the diametral axis $x - x$:

$$= y^2 dA$$
$$= (r \sin \theta)^2 \, r d\theta \, dr$$
$$= r^3 \sin^2 \theta \, d\theta \, dr$$

$\therefore$ Moment of inertia of the circle about $x - x$ is given by

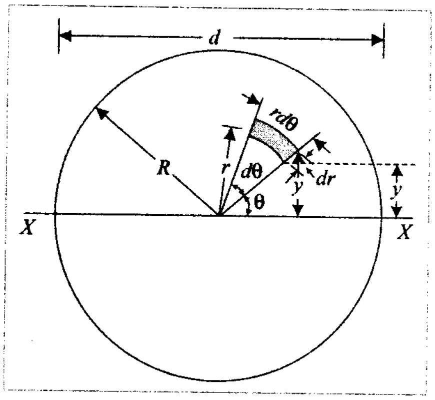


Fig. 11.8

$$I_{xx} = \int_0^R \int_0^{2\pi} r^3 \sin^2 \theta d\theta dr$$

$$= \int_0^R \int_0^{2\pi} r^3 \frac{(1 - \cos 2\theta)}{2} d\theta \, dr$$

$$= \int_0^R \frac{r^3}{2} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{2\pi} dr$$

$$= \left[\frac{r^4}{8} \right]_0^R [2\pi - 0 + 0 - 0]$$

$$= \frac{2\pi}{8} R^4$$

$$I_{xx} = \frac{\pi R^4}{4}$$

If d is the diameter of the circle, then

$$R = \frac{d}{2}$$

$$\therefore \quad I_{xx} = \frac{\pi}{4} \left(\frac{d}{2} \right)^4$$

$$I_{xx} = \frac{\pi d^4}{64} = \frac{\pi R^4}{4}$$

11.6 MOMENT OF INERTIA OF STANDARD SECTIONS

Rectangle–Referring to Fig. 11.9

(a) $I_{xx} = \dfrac{bd^3}{12}$ as derived in Sec. 11.5.

(b) $I_{yy} = \dfrac{db^3}{12}$ can be derived on the same lines.

(c) About the base AB. From parallel axis theorem,

$$I_{AB} = I_{xx} + Ay_c^2$$

$$= \frac{bd^3}{12} + bd\left(\frac{d}{2}\right)^2, \quad \text{since } y_c = \frac{d}{2}$$

$$= \frac{bd^3}{12} + \frac{bd^3}{4}$$

$$= \frac{bd^3}{3}$$

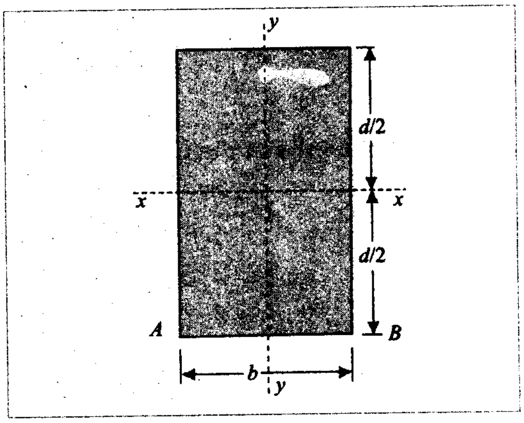


Fig. 11.9

Hollow Rectangular Section – Referring to Fig. 11.10,

Moment of inertia I_{xx}

= Moment of inertia of larger rectangle

– Moment of inertia of hollow

$$= \frac{BD^3}{12} - \frac{bd^3}{12} = \frac{1}{12}(BD^3 - bd^3)$$

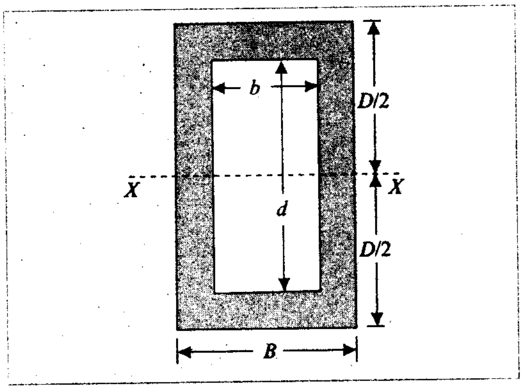


Fig. 11.10

Triangle – Referring to Fig. 11.11

(a) About the base:

As found in Sec. 11.5

$$I_{AB} = \frac{bh^3}{12}$$

(b) About centroidal axis, $x - x$ parallel to base:

From parallel axis theorem,

$$I_{AB} = I_{xx} + Ay_c^2$$

Now, y_c, the distance between the non-centroidal axis AB and centroidal axis $x - x$, is equal to $\frac{h}{3}$.

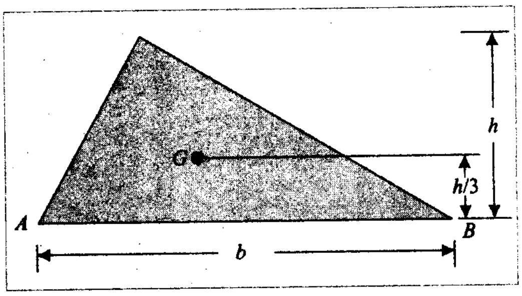


Fig. 11.11

$$\therefore \quad \frac{bh^3}{12} = I_{xx} + \frac{1}{2}bh\left(\frac{h}{3}\right)^2$$

$$= I_{xx} + \frac{bh^3}{18}$$

$$\therefore \quad I_{xx} = \frac{bh^3}{12} - \frac{bh^3}{18}$$

$$= \frac{bh^3}{36}$$

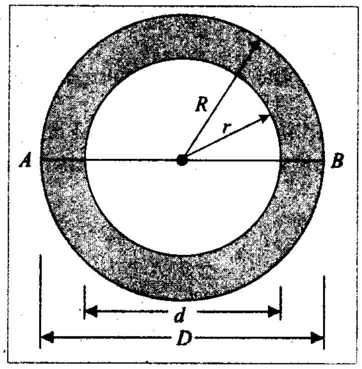


Fig. 11.12

Circle about any Diametral Axis

$$= \frac{\pi d^4}{64} \quad \text{(as found in Sec. 11.5)}$$

Hollow Circle – Referring to Fig. 11.12.

I_{AB} = Moment of inertia of a solid circle of diameter D about AB *minus* Moment of inertia of a circle of diameter d about AB. That is,

$$= \frac{\pi D^4}{64} - \frac{\pi d^4}{64}$$

$$= \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{4}(R^4 - r^4)$$

Moment of Inertia of a semicircle. (a) About Diametral Axis:

If the limit of intergration is put as 0 to π instead of 0 to 2π in the derivation of the moment of inertia of a circle about diametral axis (Ref. Art. 11.5) the moment of inertia of semicircle is obtained. It can be observed that the moment of inertia of a semicircle (Fig. 11.12(a)) about the diametral axis $AB = \frac{1}{2} \times \frac{\pi d^4}{64} = \frac{\pi d^4}{128}$

(b) *About centroidal axis x – x.* Now, the distance of centroidal axis y_c from the diametral axis is given by:

$$y_c = \frac{4R}{3\pi} = \frac{2d}{3\pi}$$

and,

$$\text{Area } A = \frac{1}{2} \times \frac{\pi d^2}{4} = \frac{\pi d^2}{8}$$

$$I_{AB} = I_{xx} + Ay_c^2$$

$$\frac{\pi d^4}{128} = I_{xx} + \frac{\pi d^2}{8} \times \left(\frac{2d}{3\pi}\right)^2$$

$$I_{xx} = \frac{\pi d^4}{128} - \frac{d^4}{18\pi}$$

$$= 0.0068598\, d^4 = 0.11 r^4$$

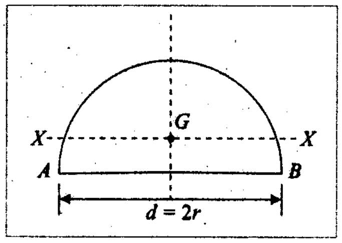


Fig. 11.12(a)

Moment of inertial of a quarter of a circle – (a) about the base:

If the limit of intergration is put as 0 to $\frac{\pi}{2}$ instead of 0 to 2π in the derivation for moment of inertial of a circle (Ref. Art. 11.5), the moment of inertia of a quarter of a circle is obtained. It can be observed that moment of inertia of the quarter of circle about the base AB

$$= \frac{1}{4} \times \frac{\pi d^4}{64} = \frac{\pi d^4}{256} = \frac{\pi d^4}{16}$$

(b) About centroidal axis x – x

Now, the distance of centroidal axis y_c from the base is given by:

$$y_c = \frac{4r}{3\pi} = \frac{2d}{3\pi}$$

and the area $$A = \frac{1}{4} \times \frac{\pi d^2}{4} = \frac{\pi d^2}{16}$$

From parallel axes theorem,

$$I_{AB} = I_{xx} + Ay_c^2$$

$$\frac{\pi d^4}{256} = I_{xx} + \frac{\pi d^2}{16}\left(\frac{2d}{3\pi}\right)^2$$

$$I_{xx} = \frac{\pi d^2}{256} - \frac{d^2}{36\pi} = 0.00343\, d^4 = 0.055 r^4$$

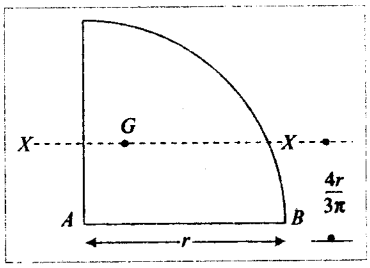


Fig. 11.12(b)

The moments of inertia of common standard sections are presented in Table 11.1.

Table 11.1 Moment of Inertia of Standard Sections

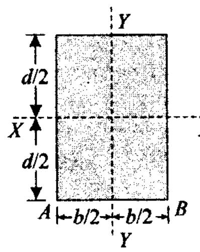
Shape	*Axis*	*Moment of Inertia*
Rectangle 	(a) Centroidal axis $x - x$	$I_{xx} = \frac{bd^3}{12}$
	(b) Centroidal axis $y - y$	$I_{yy} = \frac{db^3}{12}$
	(c) $A - B$	$I_{AB} = \frac{bd^3}{3}$

Table 11.1 Contd.

Table 11.1 Contd.

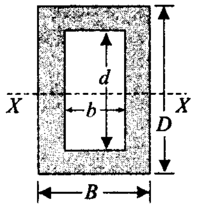
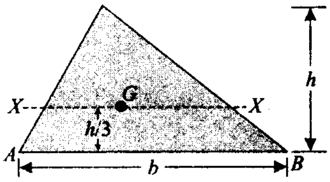
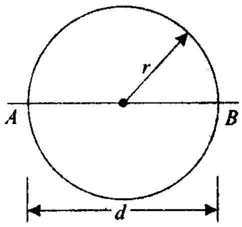
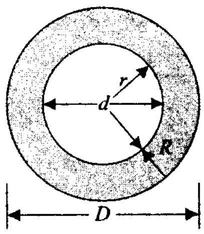
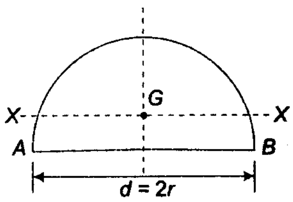
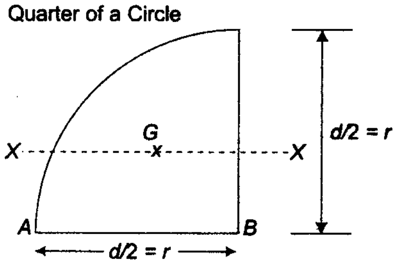
Hollow Rectangle 	Centroidal axis $x-x$	$I_{xx} = \frac{BD^3 - bd^3}{12}$
Triangle 	(a) Centroidal axis $x-x$ (b) Base AB	$I_{xx} = \frac{bh^3}{36}$ $I_{AB} = \frac{bh^3}{12}$
Circle 	Diametral axis	$I = \frac{\pi d^4}{64}$ $I = \frac{\pi r^4}{4} = \frac{\pi d^4}{64}$
Hollow Circle 	Diametral axis	$I = \frac{\pi}{64}(D^4 - d^4)$ $I = \frac{\pi}{4}(R^4 - r^4) = \frac{\pi}{64}(D^4 - d^4)$
Semi Circle 	(a) $A-B$ (b) Centroidal axis	$I_{AB} = \frac{\pi d^4}{128}$ $I_{xx} = 0.0068598\, d^4$ $= 0.11 r^4$

Table 11.1 Contd.

Table 11.1 Contd.

Quarter of a Circle 	(a) $A - B$ (b) Centroidal axis $x - x$	$I_{AB} = \dfrac{\pi d^4}{256}$ $I_{xx} = 0.00343\, d^4$ $= 0.055 r^4$

11.7 MOMENT OF INERTIA OF COMPOSITE SECTIONS

Beams and columns having composite sections are commonly used in structures. Moment of inertia of these sections about an axis can be found by the following steps:

(1) Divide the given figure into a number of simple figures.

(2) Locate the centroid of each simple figure by inspection or using standard expressions.

(3) Find the moment of inertia of each simple figure about its centroidal axis. Add the term Ay^2 where A is the area of the simple figure and y is the distance of the centroid of the simple figure from the reference axis. This gives moment of inertia of the simple figure about the reference axis.

(4) Sum up moments of inertia of al simple figures to get the moment of inertia of the composite section.

The procedure given above is illustrated below. Referring to Fig. 11.13, it is required to find out the moment of inertia of the section about axis $A - B$.

(1) The section in the figure is divided into a rectangle, a triangle and a circle. The areas of the simple figures A_1, A_2 and A_3 are calculated.

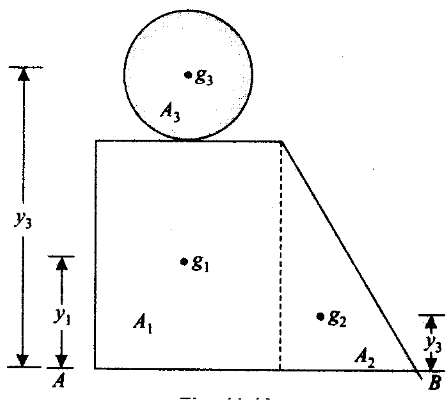


Fig. 11.13

(2) The centroids of the rectangle (g_1), triangle (g_2) and circle (g_3) are located. The distances y_1, y_2 and y_3 are found from the axis AB.

(3) The moment of inertia of the rectangle about its centroid (I_{g1}) is calculated using standard expression. To this, the term $A_1 y_1^2$ is added to get the moment of inertia about the axis AB as:

$$I_1 = I_{g1} + A_1 y_1^2$$

Similarly, the moments of inertia of the triangle ($I_2 = I_{g2} + A_2 y_2^2$) and of circle ($I_3 = I_{g3} + A_3 y_3^2$) about axis AB are calculated.

(4) The moment of inertia of the composite section about AB is given by:

$$I_{AB} = I_1 + I_2 + I_3$$

$$= I_{g1} + A_1 y_1^2 + I_{g2} + A_2 y_2^2 + I_{g3} + A_3 y_3^2 \qquad \text{Eqn. (11.9)}$$

In most engineering problems, moment of inertia about the centroidal axis is required. In such cases, first locate the centroidal axis as discussed in Art. 7.7 and then find the moment of inertia about this axis.

Referring to Fig. 11.14, first the moment of area about any reference axis, say AB, is taken and is divided by the total area of the section to locate centroidal axis $x - x$. Then the distance of centroid of individual figures y_{c1}, y_{c2}, and y_{c3} from the axis $x - x$ are determined. The moment of inertia of the composite section about the centroidal axis $x - x$ is calculated using the expression:

$$I_{xx} = I_{g1} + A_1 y_{c1}^2 + I_{g2} + A_2 y_{c2}^2 + I_{g3} + A_3 y_{c3}^2 \qquad \text{Eqn. (11.10)}$$

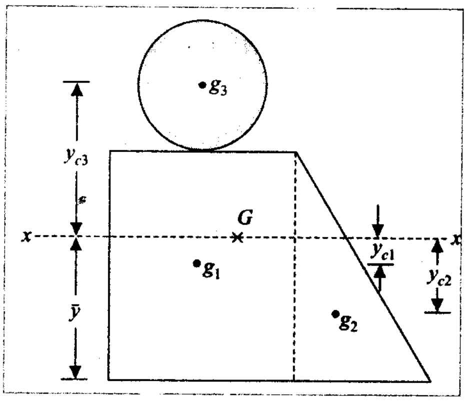


Fig. 11.14

Sometimes the moment of inertia is found about a convenient axis and then using parallel axis theorem, the moment of inertia about centroidal axis is found.

In the above example, the moment of inertia I_{AB} is found and $\bar{y}$, the distance of CG from axis AB, is calculated. Then from parallel axis theorem,

$$I_{AB} = I_{xx} + A\bar{y}^2$$

$$I_{xx} = I_{AB} - A\bar{y}^2$$

where A is the area of composite section.

Example 11.1 Determine the moment of inertia of the section shown in Fig. 11.15 about an axis passing through the centroid and parallel to the topmost fibre of the section.

Also determine moment of inertia about the axis of symmetry. Hence find radii of gyration.

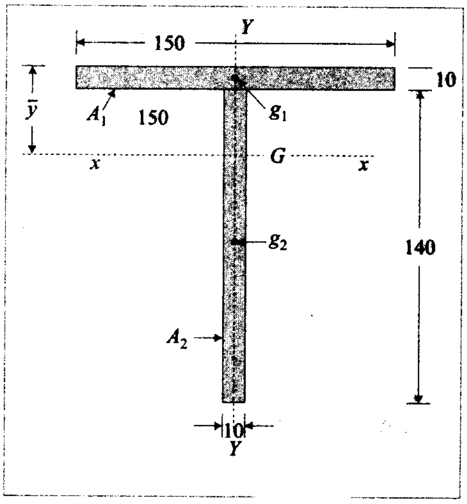


Fig. 11.15

Solution. The given composite section can be divided into two rectangles as follows:

$$\text{Area } A_1 = 150 \times 10 = 1500 \text{ mm}^2$$

$$\text{Area } A_2 = 140 \times 10 = 1400 \text{ mm}^2$$

$$\text{Total Area } A = A_1 + A_2 = 2900 \text{ mm}^2$$

Due to symmetry, centroid lies on the symmetric axis $y - y$.

The distance of the centroid from the topmost fibre is given by:

$$\bar{y} = \frac{\text{Sum of moments of the areas about the topmost fibre}}{\text{Total area}}$$

$$= \frac{1500 \times 5 + 1400(10 + 70)}{2900}$$

$$= 41.2 \text{ mm}$$

Referring to the centroidal axes $x - x$ and $y - y$, the centroid of A_1 is g_1 (0.0, 36.2) and that of A_2 is g_2 (0.0, 38.8).

Moment of inertia of the section about $x - x$ axis I_{xx} = Moment of inertia of A_1 about $x - x$ axis + moment of inertia of A_2 about $x - x$ axis.

$\therefore$

$$I_{xx} = \frac{150 \times 10^3}{12} + 1500(36.2)^2 + \frac{10 \times 140^3}{12} + 1400(38.8)^2$$

i.e.,

$$\mathbf{I_{xx} = 6.372 \times 10^6 \text{ mm}^4} \qquad \textbf{Ans.}$$

Similarly,

$$I_{yy} = \frac{10 \times 150^3}{12} + \frac{140 \times 10^3}{12} = \mathbf{2.824 \times 10^6 \text{ mm}^4} \qquad \textbf{Ans.}$$

Hence, the moment of inertia of the section about an axis passing through the centroid and parallel to the topmost fibre is 6.372×10^6 mm^4 and moment of inertia of the section about the axis of symmetry is 2.824×10^6 mm^4.

The radius of gyration is given by:

$$k = \sqrt{\frac{I}{A}}$$

$\therefore$

$$k_{xx} = \sqrt{\frac{I_{xx}}{A}}$$

$$= \sqrt{\frac{6.372 \times 10^6}{2900}}$$

$$k_{xx} = 46.9 \text{ mm} \qquad \textbf{Ans.}$$

Similarly,

$$k_{yy} = \sqrt{\frac{2.824 \times 10^6}{2900}}$$

$$k_{yy} = 31.2 \text{ mm} \qquad \textbf{Ans.}$$

Example 11.2 Determine the moment of inertia of the *L*-section shown in Fig. 11.16 about its centroidal axes parallel to the legs. Also find the polar moment of inertia.

Solution. The given section is divided into two rectangles A_1 and A_2.

$$\text{Area } A_1 = 125 \times 10 = 1250 \text{ mm}^2$$

$$\text{Area } A_2 = 75 \times 10 = 750 \text{ mm}^2$$

$$\text{Total Area} = 2000 \text{ mm}^2$$

First, the centroid of the given section is to be located.

Two reference axes (1)–(1) and (2)–(2) are chosen as shown in Fig. 11.16.

The distance of centroid from the axis (1)–(1)

$$= \frac{\text{Sum of moments of areas } A_1 \text{ and } A_2 \text{ about (1) - (1)}}{\text{Total area}}$$

i.e., $$\bar{x} = \frac{1250 \times 5 + 750\left(10 + \frac{75}{2}\right)}{2000}$$

$$= 20.94 \text{ mm}$$

Similarly, the distance of the centroid from the axis (2)–(2)

$$= \bar{y} = \frac{1250 \times \frac{125}{2} + 750 \times 5}{2000}$$

$$= 40.94 \text{ mm}$$

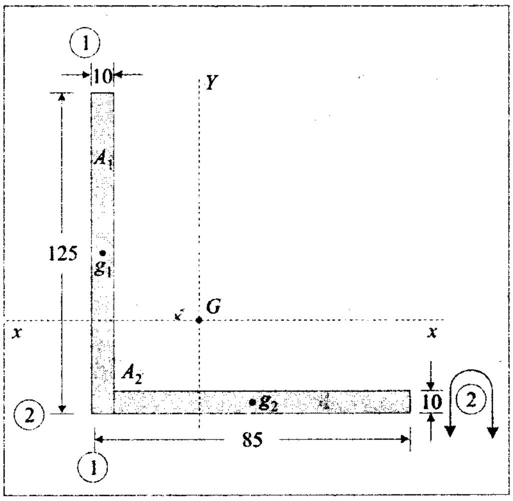


Fig. 11.16

With respect to the centroidal axes x–x and y–y, the centroid of A_1 is g_1 (15.94, 21.56) and that of A_2 is g_2 (26.56, 35.94).

$\therefore$ I_{xx} = Moment of inertia of A_1 about x–x axis + Moment of inertia of A_2 about x–x axis

$$\therefore \quad I_{xx} = \frac{10 \times 125^3}{12} + 1250 \times 21.56^2 + \frac{75 \times 10^3}{12} + 750 \times 39.94^2$$

i.e., $$\mathbf{I_{xx} = 3.4113 \times 10^6 \text{ mm}^4} \qquad \textbf{Ans.}$$

Similarly,

$$I_{yy} = \frac{125 \times 10^3}{12} + 1250 \times 15.94^2 + \frac{10 \times 75^3}{12} + 750 \times 26.56^2$$

i.e., $$\mathbf{I_{yy} = 1.2086 \times 10^6 \text{ mm}^4} \qquad \textbf{Ans.}$$

Polar moment of inertia $= I_{xx} + I_{yy}$

$$= 3.4113 \times 10^6 + 1.2086 \times 10^6$$

$$\mathbf{I_{zz} = 4.6199 \times 10^6 \text{ mm}^4} \qquad \textbf{Ans.}$$

Example 11.3 Determine the moment of inertia of the symmetric *I*-section shown in Fig. 11.17 about its centroidal axes $x - x$ and $y - y$.

Also, determine moment of inertia of the section about a centroidal axis perpendicular to $x - x$ axis and $y - y$ axis.

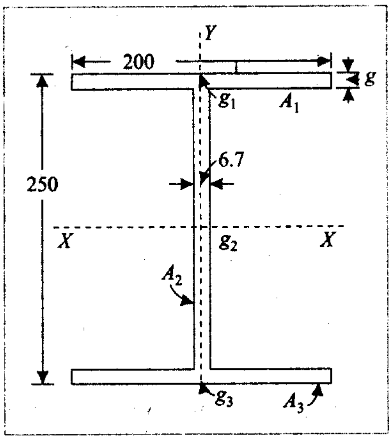


Fig. 11.17

Solution. The section is divided into three rectangles A_1, A_2, and A_3.

$$\text{Area, } A_1 = 200 \times 9 = 1800 \text{ mm}^2$$

$$\text{Area } A_2 = (250 - 9 \times 2) \times 6.7 = 1554.4 \text{ mm}^2$$

$$\text{Area } A_3 = 200 \times 9 = 1800 \text{ mm}^2$$

$$\text{Total Area } A = 5154.4 \text{ mm}^2$$

The section is symmetrical about both $x - x$ and $y - y$ axes. Therefore, its centroid will coincide with the centroid of rectangle A_2.

With respect to the centroidal axes $x - x$ and $y - y$, the centroid of rectangle A_1 is g_1 (0.0, 120.5), that of A_2 is g_2 (0.0, 0.0) and that of A_3 is g_3 (0.0, 120.5).

$$I_{xx} = \text{Moment of inertia of } A_1 + \text{Moment of inertia of } A_2 + \text{Moment of inertia of } A_3 \text{ about } x - x \text{ axis}$$

$$I_{xx} = \frac{200 \times 9^3}{12} + 1800 \times 120.5^2 + \frac{6.7 \times 232^2}{12} + 0 + \frac{200 \times 9^3}{12} + 1800(120.5)^2$$

$$\mathbf{I_{xx} = 59.2692 \times 10^6 \text{ mm}^4}$$ **Ans.**

Similarly,

$$I_{yy} = \frac{9 \times 200^3}{12} + \frac{232 \times 6.7^3}{12} + \frac{9 \times 200^3}{12}$$

$$\mathbf{I_{yy} = 12.0058 \times 10^6 \text{ mm}^4}$$ **Ans.**

Moment of inertia of the section about a centroidal axis perpendicular to $x - x$ and $y - y$ axes is nothing but polar moment of inertia, and is given by:

$$I_{zz} = I_{xx} + I_{yy}$$

$$= 59.2692 \times 10^6 + 12.0058 \times 10^6$$

$$\mathbf{I_{zz} = 71.2750 \times 10^6 \text{ mm}^4}$$ **Ans.**

Example 11.4 Compute the second moment of area of the channel section shown in Fig. 11.18 about centroidal axes $x - x$ and $y - y$.

Solution. The section is divided into three rectangles A_1, A_2 and A_3.

$$\text{Area } A_1 = 100 \times 13.5 = 1350 \text{ mm}^2$$

$$\text{Area } A_2 = (400 - 27) \times 8.1 = 3021.3 \text{ mm}^2$$

$$\text{Area } A_3 = 100 \times 13.5 = 1350.00 \text{ mm}^2$$

$$\text{Total Area } A = 5721.3 \text{ mm}^2$$

The given section is symmetric about horizontal axis passing through the centroid g_2 of the rectangle A_2. A reference axis (1)-(1) is chosen as shown in Fig. 11.18.

The distance of the centroid of the section from (1)-(1)

$$= \frac{1350 \times 50 + 3021.3 \times \frac{8.1}{2} + 1350 \times 50}{5721.3}$$

$= 25.73$ mm

With reference to the centroidal axes $x - x$ and $y - y$, the centroid of the rectangle A_1 is g_1 (24.27, 193.25), that of A_2 is g_2 (21.68, 0.0) and that of A_3 is g_3 (24.27, 193.25).

$\therefore$ I_{xx} = Moment of inertia of A_1, A_2 and A_3 about $x - x$

$$= \frac{100 \times 13.5^3}{12} + 1350 \times 193.25^2 + \frac{8.1 \times 373^3}{12} + \frac{100 \times 13.5^3}{12} + 1350 \times 193.25^2$$

$I_{xx} = \mathbf{1.359 \times 10^8}$ **mm^4**

Ans.

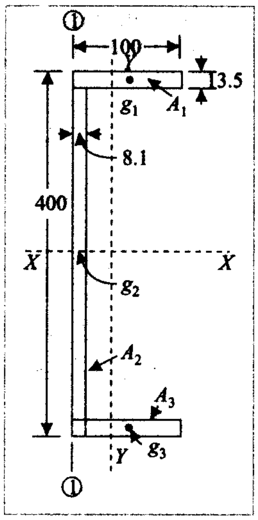


Fig. 11.18

Similarly,

$$I_{yy} = \frac{13.5 \times 100^3}{12} + 1350 \times 24.27^2 + \frac{373 \times 8.1^3}{12} + 3021.3 \times 21.68^2 + \frac{13.5 \times 100^3}{12} + 1350 \times 24.27^2$$

$I_{yy} = 52.77 \times 10^6$ mm^4

Example 11.5 Determine the polar moment of inertia about centroidal axis of the *I*-section shown in Fig. 11.19. Also determine the radii of gyration with respect to $x - x$ and $y - y$ axes.

Solution. The section is divided into three rectangles as shown in Fig. 11.19.

Area $A_1 = 80 \times 12 = 960$ mm^2

Area $A_2 = (150 - 22) \times 12 = 1536$ mm^2

Area $A_3 = 120 \times 10 = 1200$ mm^2

Total area $A = 3696$ mm^2

Due to symmetry, controid lies on axis $y - y$. The bottom fibre (1) - (1) is chosen as reference axis to locate the centroid.

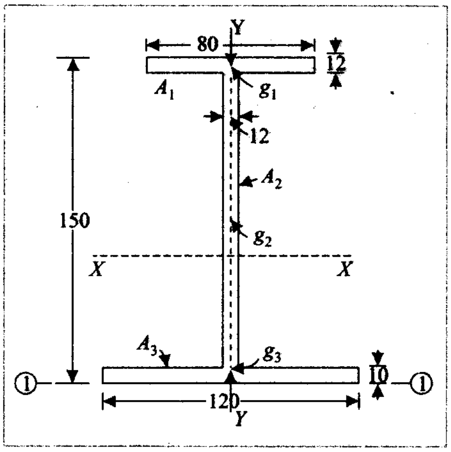


Fig. 11.19

The distance of the centroid from (1) - (1)

$$= \frac{\text{Sum of moments of the areas of the rectangles about (1) - (1)}}{\text{Total area of section}}$$

$$= \frac{960 \times (150 - 6) + 1536 \times \left(\frac{128}{2} + 10\right) + 1200 \times 5}{3696}$$

$= 69.78$ mm

With reference to the centroidal axes $x - x$ and $y - y$, the centroid of the rectangles A_1 is g_1 (0.0, 74.22), that of A_2 is g_2 (0.0, 4.22) and that of A_3 is g_3 (0.0, 64.78).

$$I_{xx} = \frac{80 \times 12^3}{12} + 960 \times 74.22^2 + \frac{12 \times 128^3}{12} + 1536 \times 4.22^2 + \frac{120 \times 10^3}{12} + 1200 \times 64.78^2$$

$$I_{xx} = 12.47 \times 10^6 \text{ mm}^4$$

$$I_{yy} = \frac{12 \times 80^3}{12} + \frac{128 \times 12^3}{12} + \frac{10 \times 120^3}{12}$$

$$= 1.9704 \times 10^6 \text{ mm}^4$$

Polar moment of inertia $= I_{xx} + I_{yy}$

$= 12.47 \times 10^6 + 1.9704 \times 10^6$

$= \mathbf{14.4404 \times 10^6}$ **Ans.**

$\therefore \quad k_{xx} = \sqrt{\frac{I_{xx}}{A}} = \sqrt{\frac{12.47 \times 10^6}{3696}}$

$=$ **58.09 mm** **Ans.**

$k_{yy} = \sqrt{\frac{I_{yy}}{A}} = \sqrt{\frac{1.9704 \times 10^6}{3696}}$

$=$ **23.09 mm** **Ans.**

Example 11.6 Determine the moment of inertia of the built-up section shown in Fig. 11.20 about its centroidal axes and $x - x$ and $y - y$.

Solution. The given composite section may be divided into simple rectangles and triangles as shown in Fig. 11.20.

Area $A_1 = 100 \times 30 = 3000 \text{ mm}^2$

Area $A_2 = 100 \times 25 = 2500 \text{ mm}^2$

Area $A_3 = 200 \times 20 = 4000 \text{ mm}^2$

Area $A_4 = \frac{1}{2} \times 87.5 \times 20 = 875 \text{ mm}^2$

Area $A_5 = \frac{1}{2} \times 87.5 \times 20 = 875 \text{ mm}^2$

Total area $A = 11250 \text{ mm}^2$

Due to symmetry, centroid lies on the axis $y - y$.

A reference axis (1) - (1) is chosen as shown in the figure.

The distance of the centroidal axis from (1) - (1)

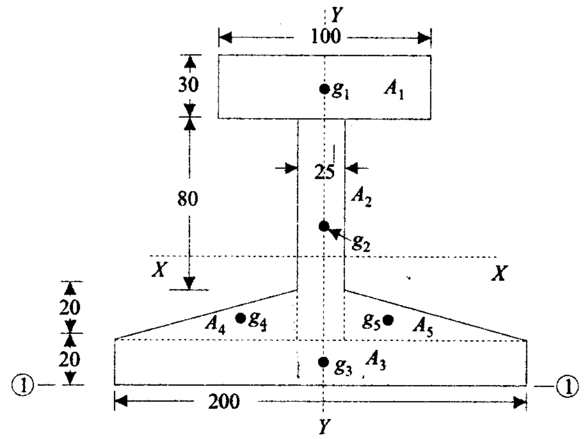


Fig. 11.20

$$= \frac{\text{Sum of moments of areas about (1) – (1)}}{\text{Total area}}$$

$$\bar{y} = \frac{3000 \times 135 + 2500 \times 70 + 4000 \times 10 + 875\left(\frac{1}{3} \times 20 + 20\right) \times 2}{11250}$$

$= 59.26$ mm

With reference to the centroidal axes x - x and y - y, the centroid of the rectangle A_1 is g_1 (0.0, 75.74), that of A_2 is g_2 (0.0, 10.74), that of A_3 is g_3 (0.0, 49.26), the centroid of triangle A_4 is g_4 (41.66, 32.59) and that of A_5 is g_5 (41.66, 32.59).

$$I_{xx} = \frac{100 \times 30^3}{12} + 3000 \times 75.74^2 + \frac{25 \times 100^3}{12} + 2500 \times 10.74^2 + \frac{200 \times 20^3}{12}$$

$$+ 4000 \times 49.26^2 + \frac{87.5 \times 20^3}{36} + 875 \times 32.59^2 + \frac{87.5 \times 20^3}{36} + 875 \times 32.59^2$$

$\mathbf{I_{xx} = 31.5434 \times 10^6}$ **mm^4** **Ans.**

$$I_{yy} = \frac{30 \times 100^3}{12} + \frac{100 \times 25^3}{12} + \frac{20 \times 200^3}{12} + \frac{20 \times 87.5^3}{36} + 875 \times 41.66^2$$

$$+ \frac{20 \times 87.5^3}{36} + 875 \times 41.66^2$$

$\mathbf{I_{yy} = 19.7451 \times 10^6}$ **mm^4** **Ans.**

Example 11.7 Determine the moment of inertia of the built-up section shown in Fig. 11.21 about an axis AB passing through the topmost fibre of the section as shown.

Solution. In this problem, it is required to find out the moment of inertia of the section about an axis AB. So there is no need to find out the position of the centroid.

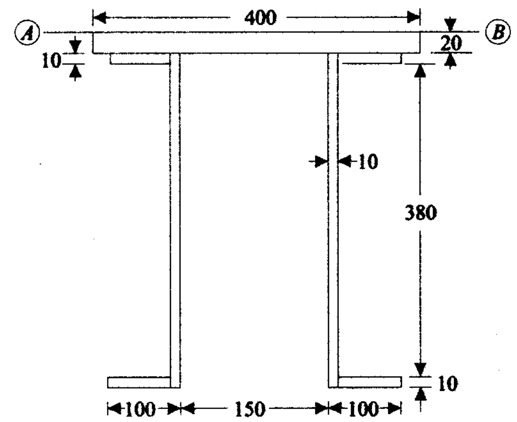


Fig. 11.21

The given section is split up into simple rectangles as shown in Fig. 11.21.

Now,

Moment of inertia about AB = Sum of moments of inertia of the rectangles about AB

$$= \frac{400 \times 20^3}{12} + 400 \times 20 \times 10^2 + \left[\frac{100 \times 10^3}{12} + 100 \times 10\,(20+5)^2\right] \times 2$$

$$+ \left[\frac{10 \times 380^3}{12} + 10 \times 380 \times (30+190)^2\right] \times 2 + \left[\frac{100 \times 10^3}{12} + 100 \times 10 \times (20+10+380+5)^2\right] \times 2$$

I_{AB} = **8.0609×10^8 mm^4** **Ans.**

Example 11.8 Calculate the moment of inertia of the built-up section shown in Fig. 11.22 about a centroidal axis parallel to AB. All members are 10 mm thick.

Solution. The built-up section is divided into six simple rectangles as shown in the figure. The distance of centroidal axis from AB

$$= \frac{\text{Sum of the moments of areas about AB}}{\text{Total Area}}$$

$$= \frac{\sum A_i y_i}{A}$$

Now,

$\Sigma A_i y_i = 250 \times 10 \times 5 + 2 \times 40 \times 10 \times (10 + 20) + 40 \times 10 \times (10 + 5)$
$+ 40 \times 10 \times 255 + 250 \times 10 \times (10 + 125)$
$= 4{,}82000$ mm^2

$A = 2 \times 250 \times 10 + 40 \times 10 \times 4$
$= 6600$ mm^2

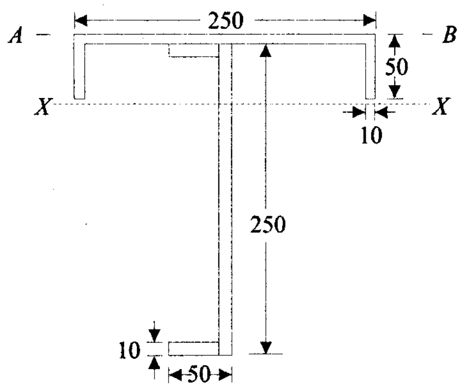


Fig 11.22

$$\therefore \quad \bar{y} = \frac{\sum A_i y_i}{A} = \frac{4{,}82000}{6600}$$

$$= 73.03 \text{ mm}$$

Now,

$$\left.\begin{array}{l}\text{Moment of inertia about the} \\ \text{centroidal axis}\end{array}\right\} = \left\{\begin{array}{l}\text{Sum of moment of inertia of the individual} \\ \text{rectangles about the centroidal axis}\end{array}\right.$$

$$= \frac{250 \times 10^3}{12} + 250 \times 10 \times (73.03 - 5)^2$$

$$+ \left[\frac{10 \times 40^3}{12} + 40 \times 10(73.03 - 30)^2\right] \times 2$$

$$+ \frac{40 \times 10^3}{12} + 40 \times 10(73.03 - 15)^2$$

$$+ \frac{10 \times 250^3}{12} + 250 \times 10(73.03 - 135)^2$$

$$+ \frac{40 \times 10^3}{12} + 40 \times 10(73.03 - 255)^2$$

$\mathbf{I_{xx} = 50.3994 \times 10^6}$ **Ans.**

Example 11.9 A built-up section of structural steel consists of a flange plate 400 mm × 20 mm, a web plate 600 mm × 15 mm and two angles 150 mm × 150 mm × 10 mm assembled to form a section as shown in Fig. 11.23. Determine the moment of inertia of the section about the horizontal centroidal axis.

Solution. Each angle is divided into two rectangles as shown in the figure.

The distance of the centroidal axis from the bottom fibres of the section

$$= \frac{\text{Sum of the moments of the areas about bottom fibres}}{\text{Total area of the section}}$$

$$= \frac{\sum A_i y_i}{A}$$

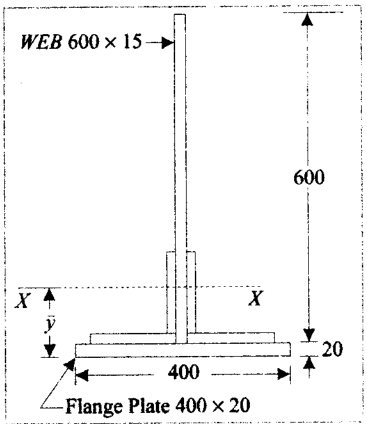


Fig. 11.23

Now, $\Sigma A_i y_i = 600 \times 15 \times \left(\frac{600}{2} + 20\right) + 140 \times 10 \times (70 + 30) \times 2$

$+ 150 \times 10 \times (5 + 20) \times 2 + 400 \times 20 \times 10$

$= 3315000 \text{ mm}^3$

$A = 600 \times 15 + 140 \times 10 \times 2 + 150 \times 10 \times 2 + 400 \times 20$

$= 22{,}800 \text{ mm}^2$

$\therefore \quad \bar{y} = \frac{\sum A_i y_i}{A} = \frac{3315000}{22{,}800}$

$= 145.39 \text{ mm}$

Moment of inertia of section about centroidal axis = Sum of the moments of inertia of the all simple figures about centroidal axis

$$= \frac{15 \times 600^3}{12} + 600 \times 15(145.39 - 320)^2 + \left[\frac{10 \times 140^3}{12} + 1400(145.39 - 100)^2\right] \times 2$$

$$+ \left[\frac{150 \times 10^3}{12} + 1500 \times (145.39 - 15)^2\right] \times 2 + \frac{400 \times 20^3}{12} + 400 \times 20 \times (145.39 - 10)^2$$

$$I_{xx} = 7.45156 \times 10^8 \text{ mm}^4 \qquad \textbf{Ans.}$$

Example 11.10 Compute the moment of inertia of the 100 mm × 150 mm rectangle shown in Fig. 11.24 about $x - x$ axis to which it is inclined at an angle $\theta = \sin^{-1}\left(\frac{4}{5}\right)$.

Solution. The rectangle is divided into four triangles as shown in the figure. [The dividing line between triangles A_1 and A_2 is parallel to $x - x$ axis].

Now,

$$\sin^{-1}\left(\frac{4}{5}\right) = 53.13°$$

From the geometry of the Fig. 11.24,

$BK = AB \sin (90° - \theta)$

$= 100 \sin (90° - 53.13°)$

$= 60$ mm

$ND = BK = 60$ mm

$\therefore \quad FD = \dfrac{60}{\sin\theta} = \dfrac{60}{\sin 53.13} = 75$ mm

$\therefore \quad AF = 150 - FD = 75$ mm

Hence $\quad FL = ME = 75 \sin \theta = 60$ mm

$AE = FC = \dfrac{AB}{\cos(90-\theta)} = \dfrac{100}{0.8} = 125$ mm

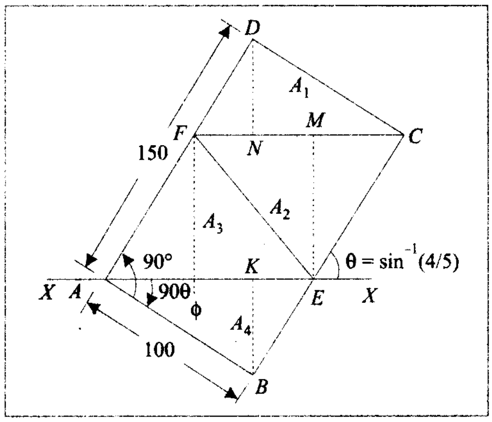


Fig. 11.24

$$\left.\begin{array}{l}\text{Moment of inertia of the section}\\ \text{about } x-x \text{ axis}\end{array}\right\} = \left\{\begin{array}{l}\text{Sum of the moments of inertia of individual}\\ \text{triangular areas about } x-x \text{ axis.}\end{array}\right.$$

$$= I_{DFC} + I_{FCE} + I_{FEA} + I_{AEB}$$

$$= \frac{125\times 60^3}{36} + \frac{1}{2}\times 125\times 60\times\left(60+\frac{1}{3}\times 60\right)^2$$

$$+ \frac{125\times 60^3}{36} + \frac{1}{2}\times 125\times 60\times\left(\frac{2}{3}\times 60\right)^2$$

$$+ \frac{125\times 60^3}{36} + \frac{1}{2}\times 125\times 60\times\left(\frac{1}{3}\times 60\right)^2$$

$$+ \frac{125\times 60^3}{36} + \frac{1}{2}\times 125\times 60\times\left(\frac{1}{3}\times 60\right)^2$$

$\mathbf{I_{xx} = 36 \times 10^6\ mm^4}$ **Ans.**

Example 11.11 Determine the moment of inertia and radii of gyration of the area shown in Fig. 11.25 about the base *A* - *B* and the centroidal axis parallel to *AB*.

Solution. (i) *MI* about base *AB*.

It may be obtained as moment of inertia of triangle ABC *minus* moment of inertia of rectangle about *A* - *B*.

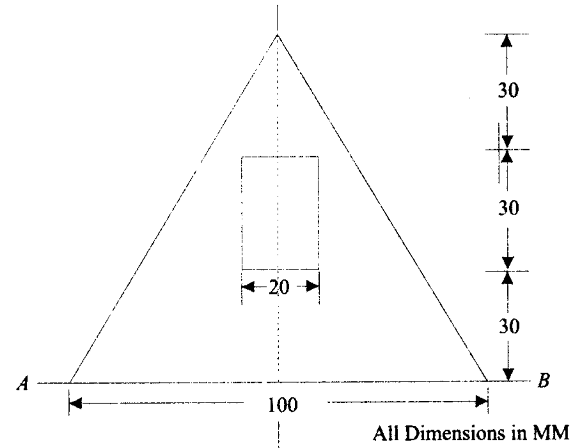


Fig. 11.25

$$I_{AB} = \frac{1}{12} \times 100 \times 90^3 - \left[\frac{1}{12} \times 20 \times 30^3 + 20 \times 30(30+15)^2\right]$$

i.e., $\mathbf{I_{AB} = 4.815 \times 10^6\ mm^4}$ **Ans.**

$\therefore$ $$k_{AB} = \sqrt{\frac{I_{AB}}{A}} = \left[\frac{4.815 \times 10^6}{\frac{1}{2} \times 100 \times 90 - 20 \times 30}\right]^{1/2} = \mathbf{35.14\ mm}$$ **Ans.**

(ii) *MI* about centroidal axis parallel to *AB*.

Let centroid be at a height $\bar{y}$ from base *AB*.

Then $$\bar{y} = \frac{\frac{1}{2} \times 100 \times 90 \times \frac{90}{3} - 20 \times 30 \times (30+15)}{\frac{1}{2} \times 100 \times 90 - 20 \times 30}$$

$= 27.69$ mm

$\therefore$ I_{xx} = *MI* of triangle about axis *xx* – *MI* of rectangle about axis *xx*

$$= \frac{1}{36} \times 100 \times 90^3 + \frac{1}{2} \times 100 \times 90(30 - 27.69)^2$$

$$- \left[\frac{1}{12} \times 20 \times 30^3 + 20 \times 30(45 - 27.69)^2\right]$$

i.e., $\mathbf{I_{xx} = 1.8242 \times 10^6\ mm^4}$ **Ans.**

$$k_{xx} = \sqrt{\frac{I_{xx}}{A}} = \frac{1.8242 \times 10^6}{\frac{1}{2} \times 100 \times 90 - 20 \times 30}$$

i.e., $\mathbf{k_{xx} = 21.63\ mm}$ **Ans.**

Example 11.12 Determine centroidal the polar moment of inertia of the plane area shown in Fig. 11.26.

Solution. Symmetric axis is taken as $y - y$ axis and let centroidal $x - x$ axis be at a distance $\bar{y}$ from top fibre as shown in the figure.

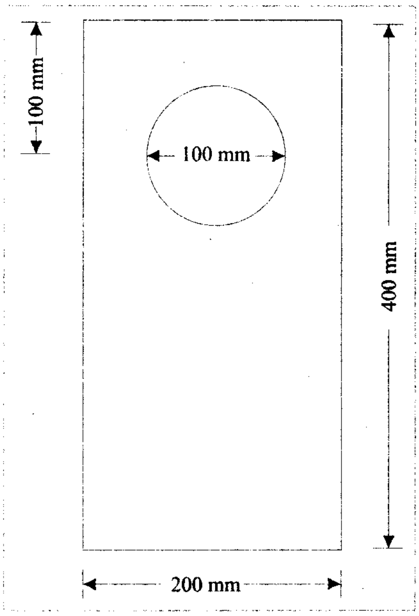


Fig. 11.26

$$\text{Total area} = 200 \times 400 - \frac{\pi \times 100^2}{4} = 72146 \text{ mm}^2$$

$$\bar{y} = \frac{\text{Moment of area about top fibre}}{\text{Total area}}$$

$$= \frac{200 \times 400 \times 200 - \pi \times \frac{100^2}{4} \times 100}{72146} = 210.9 \text{ mm}$$

$\therefore$ $I_{xx} = I_{xx}$ of rectangular area $- I_{xx}$ of circular area

$$= \frac{1}{12} \times 200 \times 400^3 + 200 + 400 \times (210.9 - 200)^2$$

$$- \left[\frac{\pi \times 100^4}{64} + \frac{\pi \times 100^2}{4} \times (210.9 - 100)^2\right]$$

Hence, $I_{xx} = 974.668 \times 10^6 \text{ mm}^4$

$$I_{yy} = \frac{1}{12} \times 400 \times 200^3 - \frac{\pi \times 100^4}{64}$$

$$= 261.758 \times 10^6 \text{ mm}^4$$

$\therefore$ Polar moment of inertia

$$I_{zz} = I_{xx} + I_{yy}$$

$$= 974.668 \times 10^6 + 261.758 \times 10^6$$

i.e., $\mathbf{I_{zz} = 1236.426 \times 10^6 \text{ mm}^4}$ **Ans.**

Important Definitions

1. *Moment of inertia* of a plane figure is the moment of moment area i.e., second moment of area.
2. Moment of inertia of an area about an axis perpendicular to its plane is called as *polar moment of inertia.*
3. *Radius of gyration* is a mathematical term defined by the relation $k = \sqrt{I/A}$, where I is moment of inertia and A is the area of plane figure.
4. *Perpendicular axis theorem* states that the moment of inertia of a given area about an axis perpendicular to the plane (polar moment of inertia) through a point 'O' is equal to the

sum of moments of inertia of that area about any two perpendicular axes through that point, the axes being in the plane of the area.

5. *Parallel axis theorem* states that the moment of inertia of an area about any axis in the plane of the area is equal to the sum of moments of inertia about the centroidal axis of the area and the product of area and square of the distance of the centroid of the area from the axis.

Important Formulae

1. $I_{xx} = \Sigma A_i\, y_i^2;\quad I_{yy} = \Sigma A_i\, x_i^2;\quad I_{zz} = \Sigma A_i\, z_i^2$
2. $k = \sqrt{I/A}$ or $I = Ak^2$
3. $I_{zz} = I_{xx} + I_{yy}$
4. $I_{AB} = I_{yy} + Ay_c^2$
5. Expressions for centroidal moment of inertia of the standard figures are as presented in Table 11.1.

Questions

1. State and prove parallel axis theorem.
2. State and prove perpendicular axis theorem.
3. Derive the expression for moment of inertia of a rectangle about its centroidal axis.
4. Derive the expression for moment of inertia of a triangle about its centroidal axis parallel to base.
5. Derive the expression for moment of inertia of a circle of diameter 'd' about its diametral axis.

Problems for Exercise

11.1. *ABCD* is a square section of sides 100 mm. Determine the ratio of moments of inertia of the section about centroidal axis parallel to a side to that about diagonal *AC*. [**Ans.** 1]

11.2. The cross-section of a rectangular hollow beam is as shown in Fig. 11.27. Determine the polar moment of inertia of the section about centroidal axes.

[**Ans.** I_{xx} = 1,05,38667 mm^4; I_{yy} = 49,06667 mm^4; I_{zz} = 1,54,45334 mm^4]

11.3. The cross-section of a prestressed concrete beam is shown in Fig. 11.28. Calculate the moment of inertia of this section about the centroidal axes parallel to and perpendicular to top edge. Also determine the radii of gyration.

[**Ans.** I_{xx} = 1,15668 × 10^{10} mm^4; k_{xx} = 231.95 mm; I_{yy} = 8.75729 × 10^9 mm^4; k_{yy} = 201.82 mm]

11.4. The strength of a 400 mm deep and 200 mm wide *I*-beam of uniform thickness 10 mm is increased by welding a 250 mm wide and 20 mm thick plate to its upper flange as shown in Fig. 11.29. Determine the moment of inertia and the radii of gyration of the composite

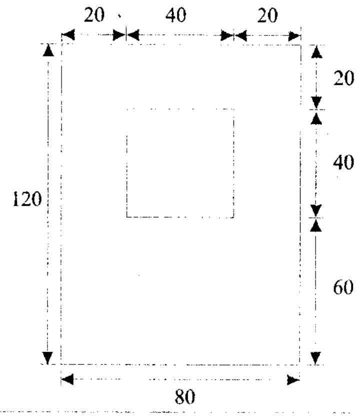


Fig. 11.27

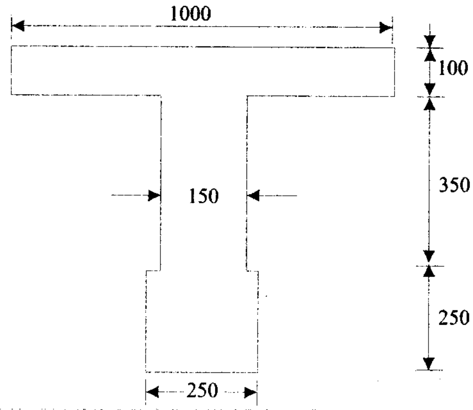


Fig. 11.28

section with respect to centroidal axes parallel to and perpendicular to the bottom edge AB.

[**Ans.** I_{xx} = 3.32393 × 10^8 mm^4; k_{xx} = 161.15 mm; I_{yy} = 3,94,06667 mm^4; k_{yy} = 55.49 mm]

11.5. The cross-section of a gantry girder is as shown in Fig. 11.30. It is made up of an I-section

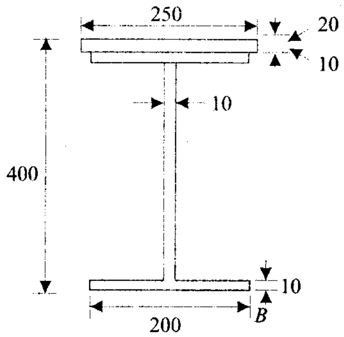


Fig. 11.29

of depth 450 mm, flange width 200 mm and a channel of size 400 mm × 150 mm. Thickness of all members is 10 mm. Find the moment of inertia of the section about the horizontal centroidal axis.

[**Ans.** I_{xx} = 4.2198 × 10^8 mm^4]

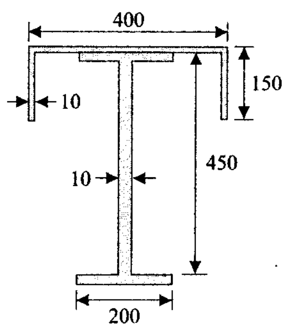


Fig. 11.30

11.6. A plate girder is made up of a web plate of size 400 mm × 10 mm, four angles of size 100 mm × 100 mm × 10 mm and cover plates of size 300 mm × 10 mm as shown in Fig. 11.31. Determine the moment of inertia about horizontal and vertical centroidal axes.

[**Ans.** I_{xx} = 5.35786 × 10^8 mm^4; I_{yy} = 6,08,50,667 mm^4]

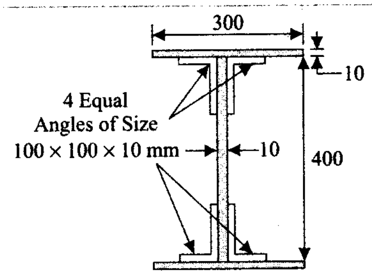


Fig. 11.31

11.7. Determine the moment of inertia of the section shown in Fig. 11.32 about the vertical centroidal axis. **[Ans.** I_{yy} = 5,03,82,857 mm^4]

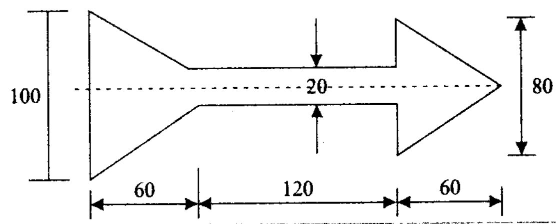


Fig. 11.32

11.8. A semi-circular cut is made in a rectangular wooden beam as shown in Fig. 11.33. Determine the polar moment of inertia of the section about the centroidal axis.

[Ans. I_{xx} = 12053349 mm^4; I_{yy} = 1,00,45631 mm^4; I_{zz} = 22098980 mm^4]

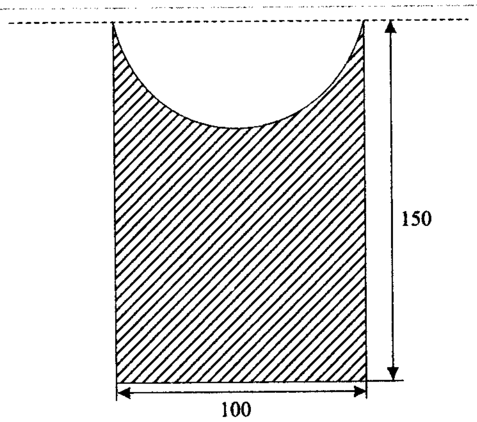


Fig. 11.33

[redacted]